

NO LIE p -ALGEBRAS OF COHOMOLOGICAL DIMENSION ONE

PASHA ZUSMANOVICH

ABSTRACT. We prove that a Lie p -algebra of cohomological dimension one is one-dimensional, and discuss related questions.

0. INTRODUCTION

A *cohomological dimension* of a Lie algebra L over a field K , denoted by $\text{cd}(L)$, is defined as the right projective dimension of the trivial L -module K , i.e., the minimal possible length of a finite projective resolution

$$(1) \quad \cdots \rightarrow P_2 \rightarrow P_1 \rightarrow P_0 \rightarrow K$$

consisting of right projective modules P_i over the universal enveloping algebra $U(L)$, or infinity if no such finite resolution exists. Since for every projective resolution (1) and every L -module M , the cohomology of the induced complex

$$0 \rightarrow M = \text{Hom}_{U(L)}(K, M) \rightarrow \text{Hom}_{U(L)}(P_1, M) \rightarrow \text{Hom}_{U(L)}(P_2, M) \rightarrow \cdots$$

of L -modules coincides with the Chevalley–Eilenberg cohomology $H^\bullet(L, M)$, L has cohomological dimension n if and only if there is an L -module M such that $H^n(L, M) \neq 0$, and one of the following equivalent conditions holds:

- (i) $H^i(L, M) = 0$ for any L -module M and any $i > n$;
- (ii) $H^{n+1}(L, M) = 0$ for any L -module M .

A similar notion may be defined for other classes of algebraic systems with good cohomology theory, e.g., for groups and associative algebras.

The Shapiro lemma about cohomology of a coinduced module implies that if S is a subalgebra of a Lie algebra L , then $\text{cd}(S) \leq \text{cd}(L)$. As cohomological dimension of the one-dimensional Lie algebra is equal to one, the cohomological dimension of any nonzero Lie algebra is ≥ 1 . In particular, the class of Lie algebras of cohomological dimension one is closed with respect to subalgebras.

Due to the standard interpretation of the second cohomology, the condition for a Lie algebra L to be of cohomological dimension one is equivalent to the condition that each short exact sequence

$$0 \rightarrow \cdot \rightarrow \cdot \rightarrow L \rightarrow 0$$

of L -modules splits. The latter condition holds for a free Lie algebra, due to its universal property, and hence a free Lie algebra (of any rank) has cohomological dimension one. The same is true for free groups and free associative algebras.

The celebrated Stallings–Swan theorem says that for groups the converse is true: a group of cohomological dimension one is free (cf., e.g., [Co]). A question by Bourbaki ([B, Chapitre II, §2, footnote to Exercice 9]) asks whether the same is true for Lie algebras, i.e., whether a Lie algebra of cohomological dimension one is free.

Feldman [Fe] answered this question affirmatively in the case of 2-generated Lie algebras. For a while, it was widely believed that the answer is affirmative in general (the author has witnessed several attempts of the proof), until Mikhalev, Umirbaev and Zolotykh constructed an example of a non-free Lie algebra of cohomological dimension one over a field of characteristic > 2 (cf. [MUZ]; note that the

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cases of characteristic zero and characteristic 2 remain widely open). This example is not a p -algebra, and at the same paper they made the following conjecture: a Lie p -algebra of cohomological dimension one is a free Lie p -algebra ([MUZ, Conjecture 2]). As stated, the conjecture is somewhat misleading, for a free Lie p -algebra is not of cohomological dimension one: its cohomological dimension is equal to infinity. Indeed, for any nonzero element x of such an algebra, the elements $x, x^{[p]}, x^{[p]^2}, \dots$ span an infinite-dimensional abelian subalgebra, whose cohomological dimension is equal to infinity (cf. Lemma 1 below).

This conjecture may be repaired in two ways. First, one may merely ask about description of Lie p -algebras of cohomological dimension one. A (trivial) answer to this question is given in §1: such algebras are one-dimensional. Another possibility is to replace cohomological dimension with *restricted* cohomological dimension; this is discussed in §2. Also, §1 contains auxiliary results and conjectures related to the old Jacobson conjecture about periodic Lie p -algebras, and to the problem of description of Lie algebras all whose proper subalgebras are one-dimensional.

1. LIE p -ALGEBRAS OF COHOMOLOGICAL DIMENSION ONE, ALMOST-PERIODIC ALGEBRAS, AND ALGEBRAS WITH ONE-DIMENSIONAL SUBALGEBRAS

The following lemma is elementary but useful.

Lemma 1.

- (i) *Cohomological dimension of an abelian Lie algebra is equal to its dimension.*
- (ii) *Cohomological dimension of the two-dimensional nonabelian Lie algebra is equal to 2.*

Proof. It is clear that cohomological dimension of a Lie algebra does not exceed its dimension.

- (i) For an abelian Lie algebra, we have $H^n(L, K) = (\wedge^n L)^*$ for any n ($*$ denotes the dual vector space).
- (ii) Let L be the two-dimensional nonabelian Lie algebra with a basis $\{x, y\}$, $[x, y] = x$. For an one-dimensional module Kv with an L -action $x \bullet v = 0, y \bullet v = -v$, we have $\dim H^2(L, Kv) = 1$. \square

Corollary. *A Lie algebra of cohomological dimension one does not contain a two-dimensional subalgebra.*

Theorem. *A Lie p -algebra of cohomological dimension one is one-dimensional.*

Proof. Let L be a Lie p -algebra of cohomological dimension one. For any $x \in L$, we have $[x, x^{[p]}] = 0$, and by Corollary to Lemma 1,

$$(2) \quad x^{[p]} = \lambda(x)x$$

for some $\lambda(x) \in K$.

Suppose L is of dimension > 1 , and pick two linearly independent elements $x, y \in L$. By [Fe], the subalgebra of L generated by x, y is free. But according to (2), $(\text{ad } x)^p(y) = \lambda(x)[y, x]$, a contradiction. \square

Let us now reflect on the condition (2). This condition reminds of various conditions on the p -map studied by Jacobson and others. The major open problem in this area is the conjecture of Jacobson that a periodic Lie p -algebra is abelian (cf. [J, Chapter V, Exercise 16]). Recall that a Lie algebra L is called *periodic* if for any $x \in L$ there is integer $n(x) > 0$ such that $x^{[p]^{n(x)}} = x$. The strongest result toward this conjecture belongs to Premet: a periodic *finite-dimensional* Lie algebra is abelian ([P, Corollary 1]).

Generalizing the condition of periodicity, let us call a Lie p -algebra L *almost periodic*, if for any $x \in L$, there is an integer $n(x) > 0$ and an element $\lambda(x) \in K$ such that

$$(3) \quad x^{[p]^{n(x)}} = \lambda(x)x.$$

The elements for which $\lambda(x) = 0$, i.e., $x^{[p]^{n(x)}} = 0$, will be called *p -nilpotent*.

Proposition 1. *Let L be an almost periodic Lie p -algebra of dimension > 1 over an algebraically closed field, with all $n(x)$'s bounded. Then L contains a nonzero p -nilpotent element.*

Note some other related results connecting properties of Lie (p -)algebras and its elements:

- (i) Chwe proved in [Ch2] that a Lie p -algebra over an algebraically closed field with a nondegenerate p -map is abelian.
- (ii) Farnsteiner investigated in [Fa1] Lie p -algebras in which some power $[p]^n$ of the p -map is p^n -semilinear. The condition (3) is somewhat reminiscent of semilinearity (in some sense stronger, in some sense weaker).
- (iii) It is well-known that any finite-dimensional Lie algebra over an algebraically closed field contains a nilpotent element. (For Lie p -algebras, this follows from the Seligman–Jordan–Chevalley decomposition, cf., e.g., [P, Proof of Theorem 3], and for a short elementary proof valid for arbitrary Lie algebras, cf. [BI]). Lemma 1 establishes a similar result for not necessary finite-dimensional Lie algebras, but subject to a strong condition of bounded p -periodicity.

Note also that the condition of the ground field being algebraically closed cannot be dropped from the Proposition, for any non-split 3-dimensional simple Lie algebra over a field of characteristic $p > 0$ provides a counterexample: it satisfies the condition $x^{[p]} = \lambda(x)x$ for any nonzero element x , but does not have nonzero p -nilpotent elements (i.e., $\lambda(x) \neq 0$ for any $x \neq 0$).

Proof of Proposition 1. Since $n(x)$ are bounded, we may assume that

$$(4) \quad x^{[p]^n} = \lambda(x)x$$

for some fixed n (for example, by letting n to be the product of all distinct $n(x)$'s, and redenoting $\lambda(x)$'s appropriately).

Pick any two linearly independent elements $x, y \in L$, and set $\varphi_{xy}(t) = \lambda(x + ty)$, for $t \in K$. Using the well-known Jacobson binomial formula for the p -map (strictly speaking, its generalization for the n th power of the p -map – cf., e.g., [Fa1, §1]), we have

$$(5) \quad \varphi_{xy}(t)(x + ty) = (x + ty)^{[p]^n} = x^{[p]^n} + t^{p^n}y^{[p]^n} + \sum_{i=1}^{p^n-1} t^i s_i(x, y) = \lambda(x)x + t^{p^n}\lambda(y)y + \sum_{i=1}^{p^n-1} t^i s_i(x, y),$$

where $s_i(x, y)$ are certain Lie monomials in x, y . Completing x, y to a basis of L , writing $s_i(x, y)$'s as linear combinations of basis elements, and collecting all coefficients of x in (5), we get that $\varphi_{xy}(t)$ is a polynomial in t with the free term $\lambda(x)$.

Suppose that there is a pair x, y such that $\varphi_{xy}(t)$ is not constant. Since the ground field K is algebraically closed, $\varphi_{xy}(t)$ has a root ξ . This means that the nonzero element $x + \xi y$ is nilpotent.

Suppose now that for any pair $x, y \in L$, $\varphi_{xy}(t)$ is constant, i.e., $\varphi_{xy}(t) = \lambda(x)$. This means that $\lambda(x + ty) = \lambda(x)$ for any linearly independent $x, y \in L$, and any $t \in K$, and, consequently, $\lambda(x) = \lambda$ is constant. If $\lambda \neq 0$, then substituting in (4) αx instead of x , we get that $\alpha^{p^n} = \alpha$ for any $\alpha \in K$, i.e., K is a finite field, a contradiction. Hence $\lambda = 0$, and every element of L is nilpotent. \square

Proposition 1 can be used to give an alternative proof of the Theorem, not utilizing the Feldman result about 2-generated Lie algebras, albeit in the case of algebraically closed ground field only (note that, in general, the cohomological dimension may increase when extending the ground field). For that, we need another elementary lemmas.

Lemma 2. *Let x, y be two elements of a Lie algebra without two-dimensional subalgebras, such that $(\text{ad } x)^n y = 0$ for some n . Then x, y are linearly dependent.*

Proof. Applying repeatedly the condition of absence of two-dimensional subalgebras, we can lower the degree n . Indeed, $(\text{ad } x)^n y = [(\text{ad } x)^{n-1}(y), x] = 0$ implies $[(\text{ad } x)^{n-2}(y), x] = (\text{ad } x)^{n-1}(y) = \lambda x$ for some $\lambda \in K$, what, in turn, implies $\lambda = 0$. Repeating this process, we get eventually $[y, x] = 0$, and hence x, y are linearly dependent. \square

Lemma 3. *A Lie p -algebra of dimension > 1 over an algebraically closed field contains a two-dimensional subalgebra.*

This lemma may be considered as a generalization of an elementary fact that a finite-dimensional Lie algebra of dimension > 1 over an algebraically closed field contains a two-dimensional subalgebra. We do not assume finite-dimensionality, but the presence of p -structure is a condition strong enough to infer the same conclusion.

Proof. Let L be a Lie p -algebra without two-dimensional subalgebras. By the same reason as in the proof of the Theorem, L satisfies the condition (2). According to Proposition 1 (with $n(x) = 1$ for all x), L is either one-dimensional, or contains a nonzero nilpotent element. In the latter case by Lemma 2, L is one-dimensional too, a contradiction. \square

Now Lemma 3, together with Corollary to Lemma 1, provides an alternative proof of the Theorem in the case of algebraically closed ground field.

Could in this proof the condition of algebraic closedness of the ground field be removed? Note that it cannot be removed from Lemma 3, with the same counterexample as in the case of Proposition 1: a non-split 3-dimensional simple Lie algebra. This is, however, the only counterexample known to us.

Conjecture 1. *A Lie p -algebra of dimension > 3 contains a two-dimensional subalgebra.*

Note that there are several intriguing open questions about Lie algebras all whose proper subalgebras are one-dimensional. It is not known whether such infinite-dimensional algebras (Lie-algebraic analogs of Tarski's monsters in group theory constructed by Olshanskii) exist. The finite-dimensional situation is, naturally, understood much better: the combination of classification of simple Lie algebras over an algebraically closed field of positive characteristic, and the standard Galois-cohomological machinery for determining forms of algebras, immediately imply that over a perfect field of characteristic $\neq 2, 3$ any finite-dimensional Lie algebra all whose proper subalgebras are one-dimensional, is (non-split) 3-dimensional simple. However, if the ground field is not perfect (in which case the Galois-cohomological machinery is not available), or is of characteristic equal to 2 or 3 (in which case the classification of simple Lie algebras is presently absent), the question about existence of such finite-dimensional algebras in dimension > 3 is open. Conjecture 1 implies that there are no such algebras in the class of Lie p -algebras (both finite- and infinite-dimensional).

2. LIE p -ALGEBRAS OF RESTRICTED COHOMOLOGICAL DIMENSION ONE

When speaking about cohomological dimension, we consider the category of *all* Lie algebra modules, including infinite-dimensional ones. If we restrict ourselves with, say, finite-dimensional Lie algebras and the category of finite-dimensional modules, the whole subject, both in results and methods employed, becomes quite different. In fact, we cannot longer speak about cohomological dimension, as vanishing of all cohomology in a given degree does not imply vanishing in higher degrees. A sample of results in this domain: in characteristic zero, an "almost" converse of the classical Whitehead Lemmas holds ([Z1], [Z2]), and in positive characteristic, for any degree less than the dimension of the algebra, a module with non-vanishing cohomology exists ([D] and [FS]).

Still, instead of the category of all modules we can consider a smaller subcategory of modules with a good-behaving cohomology theory: for example, restricted modules with restricted cohomology. Recall that for a Lie p -algebra L , and a bimodule M over its restricted universal enveloping algebra $u(L)$, we have

$$(6) \quad H_*^n(L, M^{\text{ad}}) \simeq \text{HH}^n(u(L), M),$$

where H_* and HH stand for the restricted cohomology of a Lie p -algebra, and Hochschild cohomology of an associative algebra, respectively, and M^{ad} is a restricted L -module structure on M defined via $x \bullet m = xm - mx$ for $x \in L, m \in M$.

The definition of a *restricted cohomological dimension* of L (notation: $\text{cd}_*(L)$) repeats the definition of the ordinary cohomological dimension, with projective resolutions (1) are considered in the category of restricted modules over $u(L)$.

As in the unrestricted case, Shapiro's lemma for restricted cohomology implies that the restricted cohomological dimension does not increase when passing to subalgebras. In particular, a subalgebra of a Lie p -algebra of restricted cohomological dimension one is of restricted cohomological dimension one or zero. A free Lie p -algebra has restricted cohomological dimension one.

If L is a finite-dimensional *torus*, i.e., an abelian Lie p -algebra such that any element $x \in L$ is a linear combination of its p -powers $x^{[p]^k}$, $k = 1, 2, \dots$ (over a perfect field this is equivalent to the condition that L consists of p -semisimple elements), then $u(L)$ is a commutative semisimple algebra, and hence $\text{cd}_*(L) = 0$. Conversely, the main theorem of [Hoch] (cf. also [S, Satz 10] and [Fa2, Theorem 3.1]) amounts to saying (in a different terminology) that any finite-dimensional Lie p -algebra of restricted cohomological dimension zero is a torus. Moreover, according to [W, Theorem 9.2.11], the restricted cohomological dimension zero implies finite-dimensionality, so Lie p -algebras of restricted cohomological dimension zero are exactly finite-dimensional tori.

Existence of nontrivial Lie p -algebras of restricted cohomological dimension zero allows, by the extension procedure, to get new Lie p -algebras of restricted cohomological dimension one out of old ones.

Lemma 4. *Let I be a p -ideal of a Lie p -algebra L . Then:*

- (i) $\text{cd}_*(L) \leq \text{cd}_*(I) + \text{cd}_*(L/I) + 1$;
- (ii) if $\text{cd}_*(I) = 0$, then $\text{cd}_*(L) = \text{cd}_*(L/I)$;
- (iii) if $\text{cd}_*(L/I) = 0$, then $\text{cd}_*(L) = \text{cd}_*(I)$.

Part (i) is a restricted analogue of [BK, Theorem 3.11.9][†].

Proof. This follows immediately from the Lyndon–Hochschild–Serre spectral sequence converging to $\text{HH}^{s+t}(u(L), M)$ and having the E_2 term

$$E_2^{st} = \text{HH}^s(u(L/I), \text{HH}^t(u(I), M))$$

(here M is an arbitrary $u(L)$ -module). If $\text{HH}^t(u(I), M) = 0$ for any $t > n$, and $\text{HH}^s(u(L/I), M) = 0$ for any $s > m$, then $E_2^{st} = 0$ for any $s + t > n + m + 1$, and hence $\text{HH}^i(u(L), M) = 0$ for any $i > n + m + 1$, what proves (i).

To prove (ii), note that $\text{cd}_*(I) = 0$ implies that the only non-vanishing E_2 terms are E_2^{s0} , the spectral sequences stabilizes at E_2 , and $\text{HH}^n(u(L), M) \simeq E_2^{n0} = \text{HH}^n(u(L/I), M^I)$. Since any restricted L/I -module can be lifted to a restricted L -module by letting I act trivially, the desired equality follows.

Part (iii) is established similarly: the condition $\text{cd}_*(L/I) = 0$ implies that the only non-vanishing E_2 terms are E_2^{0t} , and hence $\text{HH}^n(u(L), M) \simeq \text{HH}^n(u(I), M)^{L/I}$. The latter isomorphism implies $\text{cd}_*(L) \leq \text{cd}_*(I)$. On the other hand, $\text{cd}_*(I) \leq \text{cd}_*(L)$, what implies the desired equality. \square

Parts (ii) and (iii) of Lemma 4 show in particular, that extending a Lie p -algebra of restricted cohomological dimension zero by a Lie p -algebra of restricted cohomological dimension one, or, vice versa, a Lie p -algebra of restricted cohomological dimension one by a Lie p -algebra of restricted cohomological dimension zero, we get a Lie p -algebra of cohomological dimension one. As any extension of a Lie p -algebra of restricted cohomological dimension ≤ 1 splits, any algebra which can be obtained starting from a free Lie p -algebra by successively applying such extensions, has the following form:

$$(7) \quad (\dots((\mathcal{L} \rtimes T_1) \rtimes T_2) \dots) \rtimes T_n,$$

where \mathcal{L} is a free Lie p -algebra, T_1, \dots, T_n are finite-dimensional tori, and each symbol \rtimes stands either for \rtimes (action of the left-hand side on the right-hand side), or for \ltimes (action of the right-hand side on the left-hand side).

Conjecture 2. *Any Lie p -algebra of restricted cohomological dimension one is of the form (7).*

[†] The formulation of [BK, Theorem 3.11.9] contains an obvious typo: the summand “+1” at the right-hand side of the inequality, similarly with the inequality in Lemma 4(i), is missing.

In particular, this conjecture implies that a Lie p -algebra of restricted cohomological dimension one has a free Lie p -subalgebra of finite codimension.

Let us establish some facts about Lie p -algebras of restricted cohomological dimension one, providing a (limited) evidence in support of the conjecture.

The following fact was established in [Ch1, Theorem 5.1] using a not entirely trivial result from homological algebra due to Kaplansky. We give an alternative, more elementary proof – a mere reformulation of known (and easy) results about cohomology of commutative associative algebras.

Lemma 5. *A restricted cohomological dimension of a finite-dimensional Lie p -algebra is either zero or infinity.*

Proof. Let L be a finite-dimensional Lie algebra of restricted cohomological dimension > 0 , i.e. not a torus. Since L is not a torus, there is $x \in L$ satisfying the relation of the form

$$(8) \quad \lambda_1 x^{[p]} + \lambda_2 x^{[p]^2} + \dots + \lambda_n x^{[p]^n} = 0$$

for some $n \geq 1$ and $\lambda_1, \lambda_2, \dots, \lambda_n \in K$, $\lambda_n \neq 0$. For the p -subalgebra $(x)_p$ generated by x , we have $u((x)_p) \simeq K[x]/(f)$, where the polynomial f is obtained from the left-hand side of (8) by replacing p -powers in a Lie p -algebra by the ordinary p -powers in a polynomial algebra: $f(t) = \lambda_1 t^p + \lambda_2 t^{p^2} + \dots + \lambda_n t^{p^n}$.

The Hochschild cohomology of quotients of polynomial algebras is well understood, cf., e.g., [Hol] and references therein. In particular, in [Hol, Proposition 2.2] a periodic free resolution of such algebras is constructed, from which it follows that the complex computing the Hochschild cohomology of $K[x]/(f)$ is of the form

$$K[x]/(f) \xrightarrow{0} K[x]/(f) \xrightarrow{f'} K[x]/(f) \xrightarrow{0} K[x]/(f) \xrightarrow{f'} \dots$$

Since f' (the formal derivative of f) vanishes, $\mathrm{HH}^n(K[x]/(f), K[x]/(f))$ does not vanish for any n . As $K[x]/(f)$ is commutative, $K[x]/(f)^{\mathrm{ad}}$, as an $(x)_p$ -module, is the direct sum of p^n copies of the trivial $(x)_p$ -module K , and due to isomorphism (6), $\mathrm{H}_*^n((x)_p, K)$ is nonzero for any n . Consequently, the restricted cohomological dimension of $(x)_p$, and thus of L , is equal to infinity. \square

Proposition 2. *A p -subalgebra of a Lie p -algebra of finite restricted cohomological dimension is either a torus, or is infinite-dimensional.*

Proof. Follows from Lemma 5. \square

In particular, in a Lie p -algebra L of finite restricted cohomological dimension, every nonzero p -algebraic element is p -semisimple. This can be considered as a Lie- p -algebraic analog of the well-known fact that groups of finite cohomological dimension are torsion-free (cf., e.g., [Co, p. 6, Corollary 2]).

Proposition 3. *An abelian p -subalgebra of a Lie p -algebra of restricted cohomological dimension one is either a torus, or is isomorphic to the direct sum of a torus and the free Lie p -algebra of rank one.*

Proof. Let L be an abelian subalgebra of a Lie p -algebra of restricted cohomological dimension one. The restricted cohomological dimension of L is either equal to zero, in which case L is a torus, or is equal to one. In the latter case, assume first that L does not have nonzero p -algebraic elements.

To prove that L is a free Lie p -algebra of rank one, it is enough to prove that any two commuting elements of L , say, x and y , can be represented as p -polynomials of a third element. Suppose the contrary. By Proposition 2, each of x, y generate the free Lie p -algebra of rank one, and hence the restricted universal enveloping algebra of the p -subalgebra S of L generated by x, y , is isomorphic to the polynomial algebra in two variables $K[x, y]$. The latter algebra has non-vanishing 2nd Hochschild cohomology (for example, $\mathrm{HH}^2(K[x, y], K[x, y]) \simeq \wedge^2(\mathrm{Der}(K[x, y])) \otimes_{K[x, y]} K$ by the Hochschild–Kostant–Rosenberg theorem), and reasoning as at the end of the proof of Lemma 5, we get that $\mathrm{H}_*^2(S, K)$ does not vanish, whence $\mathrm{cd}_*(L) \geq \mathrm{cd}_*(S) \geq 2$, a contradiction.

In the general case, consider the set T of all p -semisimple elements of L . Obviously, T forms a proper subalgebra, and hence a proper ideal, of L . By Lemma 4(ii), the quotient L/T is an abelian Lie p -algebra of restricted cohomological dimension one. Since L/T does not have nonzero p -algebraic elements, L/T is isomorphic to the free Lie p -algebra of rank one by above. The extension obviously splits, and the desired conclusion follows. \square

The next lemma shows that the (ordinary) cohomology of Lie p -algebras of restricted cohomological dimension one behaves in a rather peculiar way.

Lemma 6. *Let L be a Lie p -algebra of restricted cohomological dimension one, and M a restricted L -module M . Then*

$$(9) \quad H^n(L, M) \simeq \left(\left(\bigwedge^n L \right)^* \otimes M^L \right) \oplus \left(\left(\bigwedge^{n-1} L \right)^* \otimes H_*^1(L, M) \right)$$

for any $n \geq 1$.

Proof. This follows from a particular form of the Grothendieck spectral sequence relating restricted and ordinary cohomology. Namely, for a Lie p -algebra and a restricted L -module M , there is a spectral sequence with the E_2 term

$$E_2^{st} = C^t(L, H_*^s(L, M)) \simeq \left(\bigwedge^t L \right)^* \otimes H_*^s(L, M)$$

converging to $H^{s+t}(L, M)$ (cf. [FP, Proposition 5.3]; note that the standing assumption in [FP] of finite-dimensionality of algebras and modules is not relevant here; cf. also [Fa2, Theorem 4.1] and [M, Corollary 1.3]). Here $C^n(V, W) \simeq (\bigwedge^n V)^* \otimes W$ denotes, as usual, the space of skew-symmetric n -linear maps from one vector space to another.

If $H_*^s(L, M) = 0$ for $s \geq 2$, the only nonvanishing E_2 terms are E_2^{0t} and E_2^{1t} . Hence the spectral sequence stabilizes at E_2 , $H^n(L, M) \simeq E_2^{0n} \oplus E_2^{1, n-1}$ for any $n \geq 1$, and (9) follows. \square

Lemma 6 provides a yet another proof of the fact that a Lie p -algebra L of restricted cohomological dimension one is infinite-dimensional (what follows also from Lemma 5), without appealing to any computation of Hochschild cohomology. Indeed, suppose the contrary, and take in (9) $n = \dim L + 1$. Then the left-hand side and the first direct summand at the right-hand side of the isomorphism vanish, and the second direct summand is isomorphic to $H_*^1(L, M)$. Therefore, $H_*^1(L, M) = 0$ for any restricted L -module M , i.e., L is of restricted cohomological dimension zero, a contradiction.

Moreover, a stronger statement holds:

Proposition 4. *A Lie p -algebra of restricted cohomological dimension one has infinite (ordinary) cohomological dimension.*

Proof. Let L be a Lie algebra of restricted cohomological dimension one. Taking in (9) $M = K$, we get

$$H^n(L, K) \simeq \left(\bigwedge^n L \right)^* \oplus \left(\left(\bigwedge^{n-1} L \right)^* \otimes H_*^1(L, K) \right).$$

Either by Lemma 5, or by the reasoning above, L is infinite-dimensional, and thus $\bigwedge^n L$, and hence $H^n(L, K)$, does not vanish for any $n \geq 1$. \square

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSTRAVA, OSTRAVA, CZECH REPUBLIC

E-mail address: pasha.zusmanovich@osu.cz